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On the unsteady expansion of a gas into a near vacuum

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This note intends to clarify the way in which the expansion of an initially uniform gas into an ambient atmosphere degenerates into an expansion into a perfect vacuum. The starting-point is a paper by Greenspan & Butler (1962) in which certain tentative remarks are made regarding this question. We show that the flow consists of an infinite-strength contact front coinciding with the gas–vacuum interface of the perfect-vacuum expansion, driving an infinite-strength shock into the vacuum ahead of it. The gas–vacuum interface and the shock are separated at all times even though the density of the gas through which the shock travels vanishes in the vacuum limit.

1. The problem

In a paper published in 1962 Greenspan & Butler derive a centrally important result concerning the one-dimensional unsteady expansion of an initially uniform gas into a vacuum; they show that the gas–vacuum interface in such a flow moves with a constant speed independent of the geometry be it spherical, cylindrical or plane. In their conclusions they give a brief indication of how the vacuum result might be used as a first step in an iterative approach to the more general problem of the expansion into a uniform ambient gas of low density and sound speed. It is the purpose of this note to take up this point and to clarify the role of the perfect-vacuum expansion in this context.

We consider a source gas (denoted by subscript 4) enclosed at $t = 0$ within $r = L$, surrounded by a lower density ambient gas (subscript 0). The source gas is then allowed to expand into the ambient gas and the subsequent flow is studied from the viewpoint of inviscid gasdynamics. The general picture is well known; the two gases are separated by a contact front which drives a primary shock ahead of it into the ambient gas, while in general a secondary shock forms in the source gas behind the contact front. We are interested in the motion of the contact-front–primary-shock system and in particular the way in which the flow degenerates into the perfect-vacuum expansion as the ambient density and sound speed go to zero. It is clear that in this limit the contact front will become the gas–vacuum front, so to a first approximation it may be replaced by the gas–vacuum interface of the perfect-vacuum expansion, shown by Greenspan & Butler to have velocity $2a_4/(\gamma_4 - 1)$ (where γ and a are the specific heat ratio and sound speed respectively). In this note we look at what happens to the primary shock in this limit and, in particular, examine the validity of the conjecture of Greenspan & Butler that it becomes the gas–vacuum interface.

2. The solution

An indication of the answer is provided by the plane shock tube problem, for which there is an analytic solution available. In the limit $\rho_0, a_0 \to 0$, the contact front travels with speed $2a_4/(\gamma_4-1)$ and drives an infinite-strength shock into the ambient gas with speed $(\gamma_0+1)a_4/(\gamma_4-1)$. We now exploit this picture and investigate the analytic possibilities for non-planar geometry.

The equations governing the time-dependent one-dimensional flow of an inviscid adiabatic gas are

$$\left.\begin{aligned} \frac{\partial(\rho' r'^{\sigma})}{\partial t'} + \frac{\partial(\rho' u' r'^{\sigma})}{\partial r'} &= 0, \\ \frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial r'} + \frac{1}{\rho'}\frac{\partial p'}{\partial r'} &= 0, \\ \frac{\partial(p'/\rho'^{\gamma_0})}{\partial t'} + u'\frac{\partial(p'/\rho'^{\gamma_0})}{\partial r'} &= 0, \end{aligned}\right\} \tag{2.1}$$

where p', ρ' and u' have their usual meaning and σ is the geometry index. γ_0 is the ratio of the specific heats of the ambient gas and is assumed constant. In the vacuum limit the contact front moves, to a first approximation, with speed $2a_4/(\gamma_4-1)$ and hence the boundary conditions there are

$$u' = 2a_4/(\gamma_4-1) \quad \text{on} \quad r' = 2a_4 t'/(\gamma_4-1). \tag{2.2}$$

We now consider the boundary conditions at the shock front. With conditions in front denoted by subscript 0 and those behind by subscript 1, the Rankine–Hugoniot relations give

$$\left.\begin{aligned} u_1' &= \frac{2V'}{\gamma_0+1}\{1-a_0^2/V'^2\}, \\ \rho_1' &= \rho_0\left\{\frac{\gamma_0+1}{\gamma_0-1+2a_0^2/V'^2}\right\}, \\ p_1' &= \frac{2\rho_0 V'^2}{\gamma_0+1}\left\{1-\frac{(\gamma_0-1)a_0^2}{2\gamma_0 V'^2}\right\}, \end{aligned}\right\} \tag{2.3}$$

where V' is the unknown shock velocity. Now putting $\rho_1' = \rho_1\rho_0$, $p_1' = p_1\rho_0 a_4^2$, $u_1' = a_4 u_1$, $V' = a_4 V$ and taking the vacuum limit $a_0/a_4 \to 0$, we have on the shock to a first approximation

$$u_1 = 2V/(\gamma_0+1), \quad \rho_1 = (\gamma_0+1)/(\gamma_0-1), \quad p_1 = 2V^2/(\gamma_0+1), \tag{2.4}$$

i.e. the strong shock relations. On making the same scaling in (2.1) and (2.2) together with $r' = rL$ and $t' = tL/a_4$, we get

$$\left.\begin{aligned} \frac{\partial(\rho r^{\sigma})}{\partial t} + \frac{\partial(\rho u r^{\sigma})}{\partial r} &= 0, \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} &= 0, \\ \frac{\partial(p/\rho^{\gamma_0})}{\partial t} + u\frac{\partial(p/\rho^{\gamma_0})}{\partial r} &= 0, \end{aligned}\right\} \tag{2.5}$$

with

$$u = 2/(\gamma_4-1) \quad \text{on} \quad r = 2t/(\gamma_4-1). \tag{2.6}$$

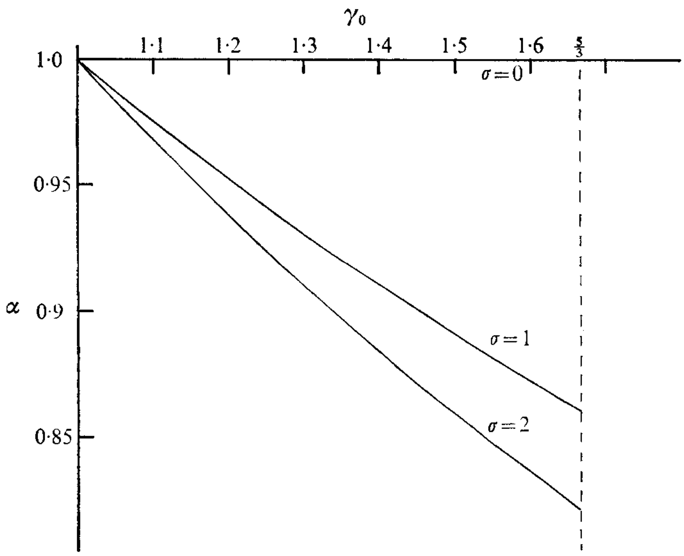

FIGURE 1. α plotted against γ_0 for plane, cylindrical and spherical flow.

We have to solve (2.5) and find V subject to (2.4), (2.6) and the initial conditions $u = 0, \rho = 1$ and $p = 0$ for $r > 1$.

This problem closely resembles that encountered in steady hypersonic small disturbance theory (for a review of this see Freeman (1965)) and can be attacked in the same way. Initially the shock moves with a velocity appropriate to the plane problem, viz.,

$$V_{t=0} = (\gamma_0 + 1)/(\gamma_4 - 1). \tag{2.7}$$

A uniformly valid large-time solution consists of an outer expansion valid near the piston and an inner expansion valid near the shock. The matching conditions and shock relations provide the boundary conditions for the inner problem of each order. The zeroth-order inner solution is the similarity solution of Taylor (1946) generalized to variable σ, γ_0 and γ_4. For $t \to \infty$, first-order matching of the large-time expansions gives

$$V_{t\to\infty} \sim \frac{\alpha(\sigma, \gamma_0)(\gamma_0 + 1)}{(\gamma_4 - 1)} + O(t^{-2(\sigma+1)}), \tag{2.8}$$

where α is a function of σ and γ_0 determined by numerical integration of two ordinary differential equations and is shown in figure 1. For $\sigma = 0, \alpha = 1$ and the error term is zero, and we have the plane problem referred to earlier.

One further point remains. It is apparent from (2.4) that the sound speed in the gas between the shock and the contact front remains of $O(1)$ in the vacuum limit. This can only be consistent with conditions immediately behind the gas–vacuum interface of the perfect-vacuum expansion, where the sound speed goes to zero, if we admit an infinite temperature jump across it. Recalling that the interface is a degenerate contact front, we see that this condition is in accord with the requirements of inviscid gasdynamics. It is in this sense that the gas–vacuum interface is an infinite-strength contact front.

In conclusion, we have shown that in the limit of vanishing ambient density and sound speed the contact front and shock driven by it have distinctly different motions. Contrary to previous conjecture the strong shock and the contact front do not coalesce.

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